stichting
mathematisch
centrum

AFDELING MATHEMATISCHE BESLISKUNDE
(DEPARTMENT OF OPERATIONS RESEARCH)

BW 63/76 DECEMBER

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# A NOTE ON THE EXPECTED PERFORMANCE OF BRANCH-AND-BOUND ALGORITHMS

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS (MOS) subject classification scheme (1970): 90B35, 90C10, 68A20

A note on the expected performance of branch-and-bound algorithms *)

by

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ABSTRACT

For many combinatorial optimization problems, the use of enumerative solution methods exhibiting a superpolynomial worst-case behaviour seems unavoidable. It is therefore of interest to investigate the expected behaviour of such methods. Polynomial-bounded expected performance has been claimed notably by M. Bellmore and J.C. Malone for their travelling salesman algorithm (*Operations Res.* 19,278-307,1766(1969)). The purpose of this brief note is to point out some inadequacies of their proof.

KEY WORDS & PHRASES: *branch-and-bound algorithm, worst-case performance, expected performance*

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*) This report will be submitted for publication elsewhere

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## 0. INTRODUCTION

A theoretical investigation of the *expected performance* of branch-and-bound algorithms is of obvious interest. Recent results on the computational complexity of many combinatorial optimization problems imply that either each of these problems can be solved within polynomial-bounded time or none of them can, and the latter alternative seems far more likely [3]. Solution methods for these problems tend to be of an enumerative nature and their *worst-case performance* is probably unavoidably superpolynomial (*e.g.*, exponential).

Up to now, polynomial-bounded expected performance has been claimed notably by M. Bellmore and J.C. Malone for their subtour-elimination approach to the asymmetric travelling salesman problem [1]. The purpose of this note is to point out some inadequacies of their proof, in the hope to encourage fresh attempts to obtain such an important result.

## 1. THE ARGUMENT OF BELLMORE AND MALONE

The argument of Bellmore and Malone can be outlined as follows. Let us consider a branch-and-bound algorithm for a minimization problem involving a *frontier search* strategy, in which subsets S are chosen for exploration in order of nondecreasing lower bounds LB(S) from the list of all subsets that have not been eliminated by further branching or by bounding considerations. If exploration of S produces a feasible solution with value equal to LB(S), then this solution is also optimal and the algorithm terminates. Suppose that such a solution is found with probability $p(S)$.

This procedure can be viewed as a statistical experiment involving a sequence of trials. The i-th trial corresponds to the exploration of the i-th chosen subset $S_i$ and results in success with probability $p_i = p(S_i)$, in which case the experiment is finished.

In the case of the subtour-elimination approach to the asymmetric travelling salesman problem, Bellmore and Malone argue that $p_1 \simeq e/n$ for large n (the number of cities) and that $p_1 \leq p_i$ for $i \geq 2$. The expected number of trials is claimed to be

$$\sum_i ip_i \Pi_{j=1}^{i-1}(1-p_j) \leq \sum_{i=1}^{\infty} ip_1(1-p_1)^{i-1} = 1/p_1 = O(n) \quad \text{for large n.}$$

At each trial, the computation of lower bounds requires the solution of $O(n)$ linear assignment problems; given an initial solution, obtained in $O(n^3)$ time, each of these is solvable in $O(n^2)$ time. Hence, the expected computational effort for the algorithm is $O(n^4)$. Computational experience is presented as confirming this result.

## 2. OBJECTIONS

It should be noted that the above argument is only valid if $p_i$ denotes the probability of success at the i-th trial under the condition that all previous trials have failed. Calculation of these conditional probabilities is not straightforward, since the trials performed at the top of the list are highly dependent.

It is not clear at all if e/n is really a lower bound on the probability $p_1$ of finding a feasible solution at the first trial, nor is it evident that $p_1$ underestimates all other (unconditional) probabilities $p_i$. Bellmore and Malone argue inconvincingly that the actual dependence works in favour of their algorithm.

Theoretically, if it could be established that $p_1 = O(n^{-c})$ for some positive constant c and that $p_1$ is a lower bound on all but a finite number of the conditional probabilities $p_i$, then the expected number of trials can be shown to be $O(n^c)$. If, in addition, the computational effort at each trial is polynomial-bounded, polynomial expected performance would follow.

As it stand, however, the argument of Bellmore and Malone is insufficiently rigorous. All that remains is a hypothesis, vaguely supported by some empirical results.

## 3. CONCLUDING REMARKS

Results on the expected performance of combinatorial tree search algorithms can only be obtained by means of careful probabilistic analysis. As a first step, such an analysis would require the specification of a probability distribution over the set of all problem instances. A natural distribution function is not always obvious, but has been suggested and explored for some problems; see, for example, the theory of random graphs as described by Erdös and Spencer [2]. Probabilistic analysis of search trees could further benefit from the well-established theory of branching processes.

Along these lines, Karp has recently arrived at various intriguing results [4]. For example, it can be proved that within a certain probabilistic model for the set covering problem any tree search algorithm having constant positive probability of finding the optimum must "almost always" explore an exponential number of nodes. On the other hand, Karp has developed a polynomial algorithm based on "bounded look-ahead plus partial backtrack" that within the same model "almost always" finds nearly optimal solutions. These and similar results require technically elaborate proofs, but could serve to explain the practical success of combinatorial algorithms whose worst-case performance is very forbidding.

## ACKNOWLEDGEMENTS

We gratefully acknowledge stimulating comments by R.M. Karp and G.L. Nemhauser.

## REFERENCES

1. M. BELLMORE AND J.C. MALONE, "Pathology of traveling-salesman subtour-elimination algorithms", *Operations Res.* 19,278-307,1766(1971).

2. P. ERDöS AND J. SPENCER, *Probabilistic Methods in Combinatorics*, Academic Press, New York, 1974.

3. R.M. KARP, "On the computational complexity of combinatorial problems", *Networks* 5,45-78(1975).

4. R.M. KARP, "The probabilistic analysis of some combinatorial search algorithms", to appear in *Proc. Symp. on Algorithms and Complexity: New Directions and Recent Results*, Academic Press, New York, 1976.